Proof of the given identity could have been obtained by the use of the ordinary complex quantities of algebra, but the above proof is given because of its greater generality and novelty.

It will be observed that the given identity as well as (1), (2), and the above 16-square theorem, all being homogeneous algebraic quadratic identities, may be given geometric interpretations by letting certain of the letters represent vectors and then taking the scalars of the resulting expressions.

2748 [1919, 72]. Proposed by J. B. REYNOLDS, Lehigh University.

The vertices of a triangle are (0, 0), ($2a$, 0), and ($2x$, $2y$). Where are the vertices of the triangle of least area having its vertices on the perpendicular bisectors of the sides of the given triangle and the same center of gravity as the given triangle?

SOLUTION BY A. M. HARDING, University of Arkansas.

Let $Q_1(x_1, y_1)$, $Q_2(x_2, y_2)$, $Q_3(x_3, y_3)$ be the vertices of the required triangle. Since Q_1, Q_2, Q_3, are on the perpendicular bisectors of the sides of the triangle $P_1P_2P_3$, we have

$$y_1y + x_1(x - a) - x^2 + a^2 - y^2 = 0, \tag{1}$$

$$y_2y + x_2x - x^2 - y^2 = 0, \tag{2}$$

$$x_3 = a. \tag{3}$$

If the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ have the same center of gravity

$$\frac{x_1 + x_2 + x_3}{3} = \frac{2x + 2a}{3}, \qquad \frac{y_1 + y_2 + y_3}{3} = \frac{2y}{3},$$

or

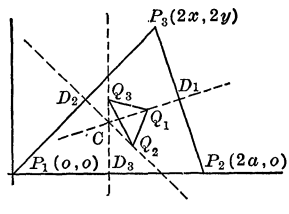
$$x_1 + x_2 = 2x + a, \tag{4}$$

$$y_1 + y_2 + y_3 = 2y. \tag{5}$$

From equations (1), (2), (4), (5), we find

$$\left.\begin{aligned} ax_1 &= -yy_3 + ax + a^2, \\ ax_2 &= yy_3 + ax, \\ ay_1 &= (x - a)y_3 + ay, \\ ay_2 &= -xy_3 + ay. \end{aligned}\right\} \tag{6}$$

The area of triangle $Q_1Q_2Q_3$ is given by



$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2a^2}\begin{vmatrix} ax_1 & ay_1 & 1 \\ ax_2 & ay_2 & 1 \\ ax_3 & ay_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2a^2}\begin{vmatrix} -yy_3 + ax + a^2, & (x - a)y_3 + ay, & 1 \\ yy_3 + ax, & -xy_3 + ay, & 1 \\ a^2, & ay_3, & 1 \end{vmatrix}$$

whence $2a\Delta = 3yy_3^2 - 2(x^2 + y^2 - ax + a^2)y_3 + a^2y$.

The area will be a minimum when $(d/dy_3)(2a\Delta) = 0$; that is, when

$$3yy_3 = x^2 + y^2 - ax + a^2.$$

It may be easily shown that the center of the circumcircle of $\Delta P_1P_2P_3$ is $C(a, y_0)$, where $yy_0 = x^2 + y^2 - ax$. Hence $3yy_3 = yy_0 + a^2$, or $y_3 = y_0/3 + a^2/3y$. The coördinates of the other vertices may now be found from equations (6).

Note. It may be shown that if the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ have the same center of gravity,

$$\frac{Q_1D_1}{P_2P_3} = \frac{Q_2D_2}{P_3P_1} = \frac{Q_3D_3}{P_1P_2},$$

where D_1, D_2, D_3 are the mid-points of the sides of Δ $P_1P_2P_3$. This property of the triangles might have been used in this problem.

Also solved by E. H. Clarke, C. E. Horne, and A. Pelletier.

2749 [1919, 72]. Proposed by C. N. SCHMALL, New York City.

In the parabola, $y^2 = 4ax$, two normals to the curve are drawn at the ends of a focal chord. Show that the area between these normals and the curve is $20a^2/(3 \sin^3 2\phi)$ where ϕ is the angle between one of the normals and the x-axis.

Solution by H. M. Roeser, Bureau of Standards, Washington, D. C.

The tangents to a parabola at the extremities of a focal chord intersect on the directrix at right angles. (Tanner and Allen, *Analytic Geometry*, page 227.) The tangents and normals will, therefore, form a rectangle of which the focal chord is a diagonal and whose area is equal to the product of the lengths of the tangents from their intersection on the directrix to the points of tangency. The area sought is the area of one of the triangular halves of the rectangle plus two-thirds of the area of the other triangle or five-sixths of the area of the rectangle.

Let m = slope of one of the normals. Then $y = mx - 2am - am^3$ is the equation of one normal and $y = -x/m + 2a/m + a/m_0$ is the equation of the other normal. $y = -x/m - am$ is the equation of one tangent, and $y = mx + a/m$ is the equation of the other tangent. The tangents intersect at the point $(x, y) \equiv [-a, a(1 - m^2)/m]$ and touch the curve at $(x, y) \equiv [am^2, -2am]$ and $(x, y) \equiv [a/m^2, 2a/m]$, respectively.

The lengths of the tangents are $l_1 = a(1 + m^2)\sqrt{1 + m^2}/m$ and $l_2 = a(1 + m^2)\sqrt{1 + m^2}/m^2$. The area sought is therefore $5l_1l_2/6 = 5a^2(1 + m^2)^3/6m^3 = 5a^2/(6 \sin^3 \phi \cos^3 \phi) = 20a^2/(3 \sin^3 2\phi)$.

Also solved by E. H. Clarke, H. H. Downing, Polycarp Hansen, C. E. Horne, Marcia L. Latham, A. Pelletier, and the Proposer.

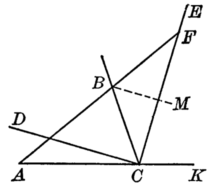
2750 [1919, 72]. Proposed by A. CAMPBELL, St. Johnsburg, Vermont.

Given the base, the sum of the sides of the triangle and the difference of the base angles, to construct the triangle.

Solution by the Proposer.

Let b, be the given base; $a + c$, the sum of the other two sides, and $\alpha = C - A$ the difference of the base angles.

On the line AK lay off $AC = b$ and at C construct an angle $ACD = \frac{1}{2}(C - A) = \frac{1}{2}\alpha$. Draw CE perpendicular to DC. With A as a center and a radius equal to $a + c$ describe an arc intersecting CE in F. Draw AF. Construct the angle BCF equal to the angle BFC. Then the triangle ABC is the required triangle.



For, triangle BCF is an isosceles triangle having its base angles equal by construction. Hence, $BC = BF$, and, therefore, $AB + BC = AF$.

Also, angle CBF = angle A + angle C, or angle MBC (BM being the bisector of angle CBF) $= \frac{1}{2}$ angle $CBF = \frac{1}{2}$(angle A + angle C) = angle BCD = angle BDC = angle A + angle DCA; whence angle $DCA = \frac{1}{2}$(angle C − angle A) $= \frac{1}{2}\alpha$.

Also solved by C. L. Arnold, George Aquis, Mary Bejsoric, P. J. da Cunha, Chang Chih-chen, H. H. Downing, A. M. Harding, C. E. Horne, Marcia L. Latham, A. Pelletier, Marian M. Torrey, and Louis Weisner.

2751 [1919, 72]. Proposed by ENOS E. WITMER, Senior in Franklin and Marshall College.

Investigate the problem of solving the equation

$$x^4 + ay^4 = w^2 + av^2. \qquad (1)$$

Carmichael's *Diophantine Analysis*, Problem 18, p. 54.